1.) choose reference frame - coordinate system $(x, y \rightarrow e_r, e_\theta)$
- origin

2.) Free Body Diagram - constraints

3.) Linear Momentum Balance (LMB), Angular Momentum Balance (AMB)

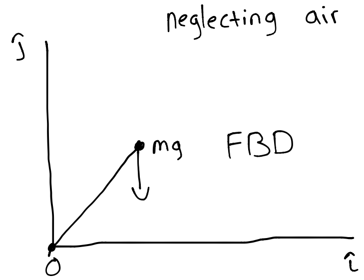
$\vec{F} = m\frac{d^2\vec{r}}{dt^2}$

4.) Differential Equation
- hand
- Matlab

Basic Ballistics

Reference Frame $\rightarrow$ Cartesian

neglecting air resistance!



$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} = -mg\hat{j}$

$\ddot{\vec{r}} = -g\hat{j}$

$\dot{\vec{r}} = \vec{V}_0 - gt\hat{j}$

$\vec{r} = \vec{r}_0 + \vec{V}_0 t - \frac{1}{2}gt^2\hat{j}$

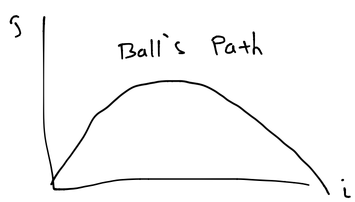
Initial Conditions

$\vec{r}_0 = 0, \ \vec{V}_0 = V_0\cos\theta\,\hat{i} + V_0\sin\theta\,\hat{j}$

$$\boxed{\vec{r} = V_0\cos\theta\, t\,\hat{i} + \left(V_0\sin\theta\, t - \frac{1}{2}gt^2\right)\hat{j}}$$

Break down $\vec{r}$ into its components

$\ddot{\vec{r}} \rightarrow \begin{matrix}\ddot{x} = 0 \\ \ddot{y} = -g\end{matrix}$ $\qquad$ $r \rightarrow \begin{matrix}x = V_0\cos\theta t \\ y = V_0\sin\theta t - \frac{1}{2}gt^2\end{matrix}$



45° to maximize range

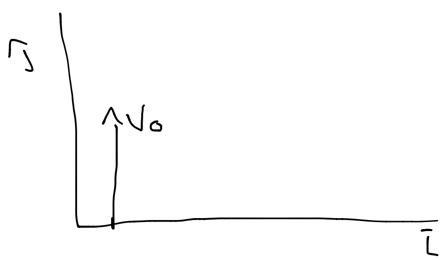
Air Drag

$F_{\text{Linear}} = cv$ — Linear Viscous Drag; opposite to direction of velocity

$F_{\text{quadratic}} = cv^2$ — quadratic viscous drag; opposite to direction of velocity $\longrightarrow$ Imparting momentum to the air

$c = \frac{1}{2}\rho A v^2$

Ballistics with Linear Drag



$$\vec{F} = m\vec{a} = m\ddot{y} = -c\dot{y} - mg$$

$$m\dot{v} = -cv - mg$$

$$\frac{m\,dv}{-cv - mg} = dt \quad \text{Differential Equation}$$

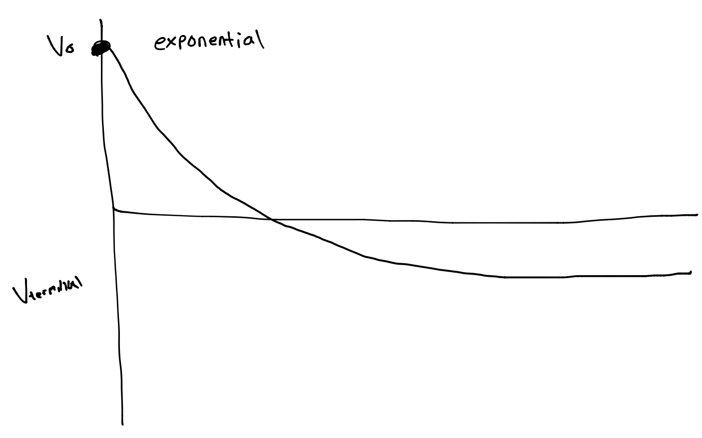
$$v(t) = -V_t + (V_t + V_0)e^{-t/\tau}$$

$V_t \rightarrow \dot{v} = 0$ — terminal velocity

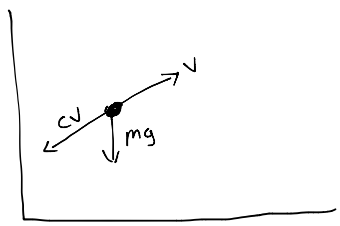
$V_t \rightarrow mg = -cv$

$V_t = -\frac{mg}{c}, \quad \tau = \frac{m}{c}$

$$v(t) = -V_t + (V_t + V_0)e^{-t/\tau}$$



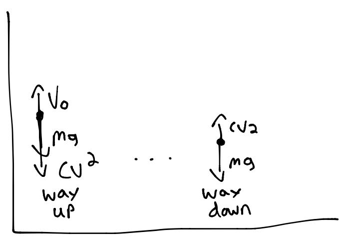
2-D



LMB: $m\frac{d^2\vec{r}}{dt^2} = -mg\hat{j} - c\vec{V}$ $\quad \vec{V} = V\hat{v} \quad (cV\hat{v})$

$$\boxed{\begin{aligned} &m(\ddot{x}\hat{i} + \ddot{y}\hat{j}) = -mg\hat{j} - c(\dot{x}\hat{i} + \dot{y}\hat{j}) \\ &m\ddot{x} = -c\dot{x} \\ &m\ddot{y} = -c\dot{y} - mg \end{aligned}}$$

Quadratic Drag

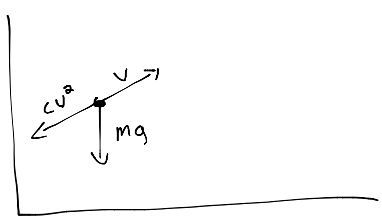


$m\dot{v} = -mg - cv^2 \quad v>0 \qquad m\dot{v} = -mg + cv^2 \quad v<0$

$\ddot{y} = -cv^2 - mg \qquad \ddot{y} = cv^2 - mg$

$v_t = -\sqrt{\frac{mg}{c}}$

2-D



LMB: $m\frac{d^2\vec{r}}{dt^2} = -mg\hat{j} + c|\dot{r}|^2\hat{r}$

$m\ddot{r} = -mg\hat{j} - cv^2\hat{\vec{v}} = -mg\hat{j} - cv^2\frac{\vec{v}}{|\vec{v}|}, \quad |\vec{v}| = v$

$m\ddot{\vec{r}} = -mg\hat{j} - cv\vec{v}$

$$\boxed{\begin{aligned} &m\ddot{x} = -(c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}) \\ &m\ddot{y} = -mg - c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2} \end{aligned}}$$